Journal of Engineering Mathematics **49:** 1–18, 2004.

An infinitely large plate weakened by a circular-arc crack subjected to partially distributed loads

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Received 13 January 2003; accepted in revised form 26 May 2003

Abstract. On the basis of the complex-variable approach for the first boundary condition problems, a mapping function is proposed to transform the contour surface of a circular arc crack into a unit circle. By this mapping, direct stress integration along the contour surface can be performed for the case when uniform tractions are applied on part of the crack edge. General complex stress functions are obtained by evaluating the Cauchy integral for the governing boundary equation. After the obtained stress functions are differentiated with respect to a reference angle in the mapped plane, the general complex stress functions for the circular-arc crack problem, when concentrated loads are applied on the crack surface, can be obtained. The importance of this solution lies in its general applicability to crack problems with arbitrary loading.

Key words: circular-arc crack, complex-variable method, partial loading

1. Introduction

From past research on crack analysis, our knowledge about a crack modeled by a straight slit enables engineers to establish a reasonably accurate assessment of the fracture properties of a structure containing a flat crack. However, comparably few research works are seen to deal with the fracture problem for a curved crack. Indeed, an initially straight crack will most likely grow to form a curvilinear one under a non-uniform mixed-mode stress field. It has also been shown by Chaker and Barquins [1] that a branch crack tends to grow to form a curvilinear one under compression. Crack propagation is outside the scope of the present study and therefore no further discussion is planned in this paper about the shape of a crack after its propagation. The main purpose of this article is to provide an optional crack model that could be used to analyze the fracture of a large plate containing a crack that is not perfectly straight after its propagation.

Introducing an elegant concept of discontinuity, Muskhilishvili [2, pp. 358–361] was the first to investigate the circular-arc crack (CAC) problem under a biaxial tension condition. Thereafter, Sih *et al.* [3] gave the corresponding stress-intensity factor (SIF) through a limiting process. Most studies have been mainly devoted to dealing with the extensional problem for a curved crack. In this regard, Cotterell and Rice [4] presented a solution of the SIF of a slightly curved or kinked crack. Ioakimidis and Theocaris [5] gave a numerical CAC solution in an isotropic elastic half-plane. Also, Chen and Hasebe [6] obtained an elementary solution for multiple CAC problems. Making use of the basic theorem of the pole point and residue, Zhangh [7] attemped to solve the CAC problem with a concentrated force applied on the crack surface. However, the obtained solution is not valid due to this approach decoupling fracture modes I, II. In fact, both fracture models I, II are coupled due to the crack configuration that is defined by a parameter in the present analysis. Considering out-of-plane loads, Van

Vroonhoven [8] derived a first-order solution to the problem that a thin plate weakened by a curvilinear crack of finite size is subjected to out-of-plane bending and twisting moments. Incorporating the out-of-plane shearing effect, Shiah [9] gave a closed-form solution to the general out-of-plane fracture problem of a CAC-containing plate subjected to general remote out-of-plane loadings, including bending, twisting, and shearing. Also, Shiah *et al.* [10] gave an approximate solution to the problem that a CAC-containing large beam is subjected to in-plane bending moments at both ends.

For the present work, a mapping function is applied to investigate the fracture of an infinitely large plate weakened by a CAC on which partially distributed loads are applied. The complex-stress functions for the CAC problem are derived by the complex-variable technique and the corresponding SIFs are given through a limiting process.

In contrast with the approach of Zhang [7], the present approach is to attack the problem by considering the partial loading on the crack surface. Further, the solution is extended to consider a concentrated force applied on the crack surface. This solution can be regarded as the Green's function to formulate the solutions of other cases under arbitrary loads. The major advantage of this approach, distinguishing itself from reference [7], is that the obtained closed-form solution is also valid for the straight-crack problem without taking any limiting process. This important feature feature enables us to get a clear insight into the curvature effect of a crack on the stress and displacement fields. In contrast with past work for many other CAC problems, the elegance of the present approach indeed lies in iets integration of both solutions to the CAC and the straight-crack problem into one unique form, yet without characterizing the two distinct problems into two separate problem categories.

2. Conformal mapping

When calculating stress functions, it is expedient for the purpose of evaluating the Cauchy integral to transform the boundary contour into a unit circle. It is well known that an ellipse can be transformed into a unit circle and the ellipse can be further degenerated into a slit (with crack length 2a) through the mapping,

$$\omega(\zeta) = \frac{a}{2}(\zeta + \zeta^{-1}), \tag{1}$$

where ζ is the complex variable in the mapped plane. As a matter of fact, the above mapping can be further modified to transform a contour surface into a unit circle through the following function,

$$z = \omega(\zeta) = R\left(\frac{\zeta^2 + 1}{\zeta + \mathrm{i}c}\right), \tag{2}$$

where $R = a(1\text{-}c^2)^{-1/2}/\ 2$ and c ($0 \leqslant c < 1$)are mapping variables determining the crack configuration. When $\zeta = \xi = \mathrm{e}^{\mathrm{i}\beta}$ (where ξ is denoted as the boundary value of ζ on the unit circle ρ throughout this paper; β is the reference angle of an arbitrary point on the unit circle) is substituted in Equation (2), it can be readily shown that this contour line is nothing but an arc of a circle with radius $r_0 = a/[2c(1 - c^2)^{1/2}]$ and chord length $2a$. Particularly, if zero is chosen for the constant c, the above mapping equation is identical with Equation (1), which refers to the case of the straight-crack problem. This mapping function, considered

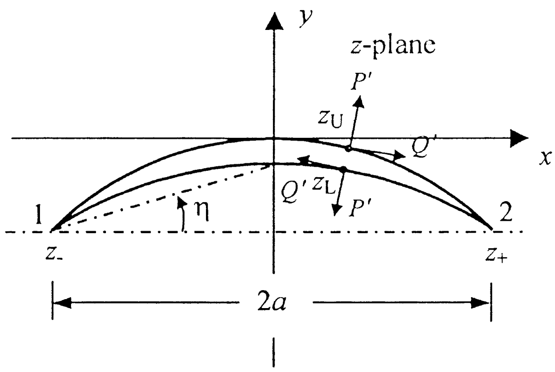

Figure 1. Concentrated loads applied on the surface of a circular-arc crack.

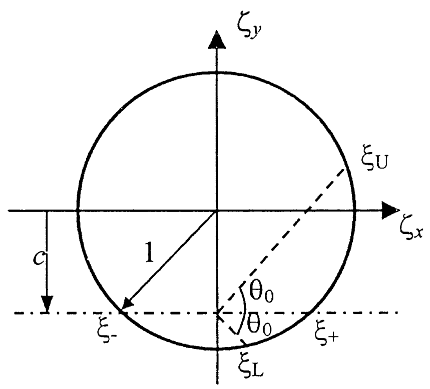

Figure 2. Transformation of a circular-arc crack in the z-plane to a unit circle in the ζ-plane.

as an extension of the particular form for the straight-crack problem, actually transforms the outside region of the contour surface of a circular arc crack onto the outside region of a unit circle in the mapped plane. Through this transformation described by Equation (2), the crack tips 1 (at z_-) and 2 (at z_+), having the coordinates

$$z_\pm = \pm a - \mathrm{i}ca/\sqrt{1-c^2}, \tag{3}$$

are eventually mapped to points ξ_- and ξ_+ in the ζ-plane, given by

$$\xi_\pm = \pm\sqrt{1-c^2} - \mathrm{i}c. \tag{4}$$

To interpret the geometrical meaning of the variable c, one may draw a triangle connecting both crack tips and the origin. By making some algebraic operations, one can readily show that this mapping variable c is given by

$$c = \sin\eta, \tag{5}$$

where the angle η is defined as shown in Figure 1. By introducing this non-dimensional angle, one may find it expedient in defining the configuration of a circular-arc crack. At one particular position of a CAC, there are two points, coinciding with each other at the same coordinate. As shown in Figure 2, the image points of z_L and z_U, where the subscripts L and U are used to denote, respectively, the lower and upper crack surface, have the following expressions,

$$\begin{aligned} \xi_L &= (-c\sin\theta_0 + \sqrt{1-c^2\cos^2\theta_0})\,\mathrm{e}^{-\mathrm{i}\theta_0} - \mathrm{i}c, \\ \xi_U &= (c\sin\theta_0 + \sqrt{1-c^2\cos^2\theta_0})\,\mathrm{e}^{\mathrm{i}\theta_0} - \mathrm{i}c, \end{aligned} \tag{6}$$

in which θ_0 is the angle measured from the reference line, introduced to facilitate the transformation. By substituting Equation (6) in Equation (2), the coordinates of the points in the z-plane are expressed as

$$z_L = z_U = 2R\cos\theta_0(-\mathrm{i}c\cos\theta_0 + \sqrt{1-c^2\cos^2\theta_0}). \tag{7}$$

As is obvious from Equation (7), corresponding points in the physical plane will always have the same coordinates, whether or not θ_0 is positive (for the upper surface) or negative (for the lower surface).

3. Partially distributed uniform loads

3.1. Complex-stress functions

As a preliminary to deriving the solution to the problem for concentrated loading, the case is first considered that the crack edge is partially loaded with uniform loads. As described earlier, the mapping $\omega(\zeta)$ given by Equation (2) is utilized to transform the unbounded domain outside a circular-arc crack (including its surface) onto the unbounded region outside the unit circle (including the circle) in the auxiliary plane. Now, consider the case that only part of the crack edge is subjected to uniform normal and tangential tractions. The surface $z_L - z_+ - z_U$ is subjected to uniform normal traction P and tangential traction Q, while the surface $z_U - z_- - z_L$ is traction-free. By use of the mapping, arbitrary points on the upper and lower surface z_U and z_L are mapped to ξ_U and ξ_L, respectively (see Figure 2). From the mapping principle, the arc $\xi_L - \xi_+ - \xi_U$ should also carry the same amount of traction corresponding to those in the z-plane. From the basic complex-variable technique, the boundary equation can be written as

$$\phi(\xi) + \frac{\omega(\xi)}{\overline{\omega'(\xi)}}\overline{\phi'(\xi)} + \overline{\psi(\xi)} = \textstyle\int_s (N_n + \mathrm{i}\,N_t)\,\mathrm{d}z_s, \tag{8}$$

where ϕ and ψ are complex stress functions that are holomorphic in S^-, where $S^- = \{\zeta \in C \mid |\zeta| > 1\}$ and continuous on ρ; N_n and N_t are used to denote the outward normal traction and tangential traction along the boundary S, respectively. In Equation (8), the integral on the right-hand side is the boundary stress integration function, denoted for brevity by f for all derivations throughout this paper. Performing Cauchy integration for each term in Equation (8), one may rewrite the boundary equation as, for $\zeta \in S^-$,

$$-\phi(\zeta) + \frac{1}{2\pi\,\mathrm{i}}\int_\rho \frac{\omega(\xi)}{\overline{\omega'(\xi)}} \cdot \frac{\overline{\phi}'(\xi^{-1})}{\xi - \zeta}\mathrm{d}\xi = \Im(\zeta) \tag{9}$$

where $\Im(\zeta)$ is used to denote the Cauchy integral of f. In arriving at the above equation, the Cauchy integral of $\overline{\psi(\xi)}$ is dropped, since the conjugate of ψ is holomorphic inside ρ. On inserting the expressions for $\omega(\xi)$, $\omega'(\xi)$, and $\overline{\phi}'(\xi^{-1})$ into Equation (9) and applying Cauchy's formulae, one may immediately obtain

$$-\phi(\zeta) - \frac{(1-c^2)^2\,\overline{\phi}'(-1/\mathrm{i}c)}{\zeta + \mathrm{i}c} = \Im(\zeta). \tag{10}$$

To solve for $\phi(\zeta)$, differentiation with respect to ζ is performed upon the conjugate form of Equation (10) and let ζ equal $-\mathrm{i}c^{-1}$ to give

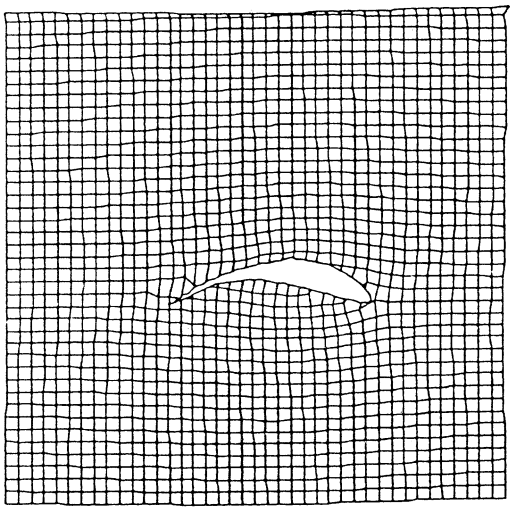

Figure 3. Deformation of a CAC-containing plate half loaded with tension ($c = 0{\cdot}3$).

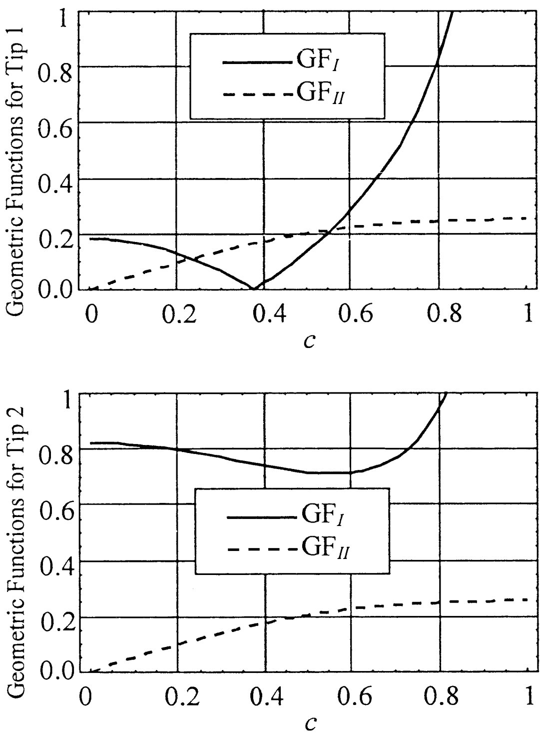

Figure 4. Variation of Geometric Functions with variable c for pure tension.

$$\overline{\phi}'(\mathrm{i}\,c^{-1}) = \frac{(1-c^2)^2\,\phi'(-\mathrm{i}\,c^{-1})}{(-\mathrm{i}\,c^{-1}+\mathrm{i}c)^2} - \overline{\Im}'(-\mathrm{i}\,c^{-1}). \tag{11}$$

Therefore, the constant $\bar{\phi}'(\mathrm{i}\,c^{-1})$ can be immediately determined by substituting Equation (11) back in Equation (10) and letting ζ equal $-\mathrm{i}c^{-1}$. In the sequel, it can be written as

$$\overline{\phi}'(\mathrm{i}\,c^{-1}) = \frac{\overline{\Im}'(\mathrm{i}\,c^{-1}) - c^2\,\Im'(-\mathrm{i}\,c^{-1})}{(c^4-1)} \tag{12}$$

As a result of substituting Equation (12) back in Equation (10), the expression for the complex stress function ϕ eventually takes the form,

$$\phi(\zeta) = -\Im(\zeta) - \frac{1-c^2}{1+c^2}\cdot\frac{c^2\,\Im'(-\mathrm{i}\,c^{-1}) - \overline{\Im}'(\mathrm{i}\,c^{-1})}{\zeta+\mathrm{i}c}. \tag{13}$$

Recall that $\Im(\zeta)$ represents the Cauchy integral of the boundary stress integration function with tractions specified along the crack surface. As long as the tractions are specified, one may directly obtain the expression for ϕ, and thus Equation (13) is considered the general expression for the stress function ϕ of the CAC problem. Similarly, the other stress function ψ can be determined by taking the Cauchy integral of the conjugate form of the boundary equation, given by Equation (8). Consequently, one may obtain

$$\psi(\zeta) = \frac{-1}{2\pi \mathrm{i}} \int_\rho \frac{\overline{f}(\xi^{-1})}{\xi - \zeta} \mathrm{d}\xi + \frac{1}{2\pi \mathrm{i}} \int_\rho \frac{\overline{\omega}(\xi^{-1})}{\omega'(\xi)} \cdot \frac{\phi'(\xi)}{\xi - \zeta} \mathrm{d}\xi. \tag{14}$$

In arriving at the above equation, the constant term $\psi(4)$ has been omitted due to the fact that no contribution of this term is made to all stress components. Also, the Cauchy integral of $\overline{\phi}(\xi^{-1})$ is dropped out, since the conjugate of $\phi(\xi)$ is holomorphic inside ρ, and therefore its Cauchy integral turns out to be zero. Suppose the uniform traction F, consisting of an outward normal constant traction -P(compression) and a tangential constant traction Q (in clockwise direction), act upon the crack edge $z_L - z_+ - z_U$ without loading at infinity. Accordingly, the traction F can be expressed as

$$\begin{aligned} F &= -P + \mathrm{i}Q \quad , \quad \text{on} \quad z_L - z_+ - z_U, \\ &= 0 \qquad\qquad , \quad \text{on} \quad z_U - z_- - z_L. \end{aligned} \tag{15}$$

For this case, the quantity $f(\xi)$, defined by $f(\xi) = \int_s (N_n + \mathrm{i}\, N_t) \mathrm{d}\, z_s$, has to be written separately for the loaded and unloaded region as

$$f(\xi) = F R \int_{\xi_-}^{\xi} \mathrm{d} \left(\frac{\zeta^2 + 1}{\zeta + i\, c} \right) = \begin{cases} F R \left(\frac{\xi^2 + 1}{\xi + \mathrm{i}c} \right) - F R \left(\frac{\xi_L^2 + 1}{\xi_L + \mathrm{i}c} \right), & \xi \in \text{arc } \xi_L - \xi_+ - \xi_U, \\ 0 & \xi \in \text{arc } \xi_U - \xi_- - \xi_L\,. \end{cases} \tag{16}$$

Since Cauchy's formulae may be applied merely to the case when the integrand is continuous along the closed path, they cannot be used for this case when loads are applied on only part of the unit circle. Instead of applying Cauchy's formulae, the Cauchy integral of $f(\xi)$, denoted by $\Im(\zeta)$ herein, is evaluated *via* direct integration. With the integration constant dropped, since it has no influence on further results, this Cauchy integral of $f(\xi)$ is found to be

$$\Im(\zeta) = \frac{F R}{2\pi \mathrm{i}} \left[\frac{c^2 - 1}{\zeta + \mathrm{i}c} \log \frac{\xi_U + \mathrm{i}c}{\xi_L + \mathrm{i}c} + \left(\frac{\zeta^2 + 1}{\zeta + \mathrm{i}c} - \frac{z_0}{R} \right) \log \frac{\xi_U - \zeta}{\xi_L - \zeta} \right], \tag{17}$$

where z_0, the point at which the tractions discontinue, is given by

$$z_0 = z_{U \backslash L} = R \left(\frac{{\xi_{U \backslash L}}^2 + 1}{\xi_{U \backslash L} + \mathrm{i}c} \right). \tag{18}$$

In Equation (18), it should be noted that both ξ_U and ξ_L have the same z_0 in the physical plane. To complete the solution to the partially loaded CAC problem, differentiation of $\Im(\zeta)$ with respect to ζ is performed to yield

$$\Im'(\zeta) = \frac{FR}{2\pi \mathrm{i}}\left[\frac{1-c^2}{(\zeta+\mathrm{i}c)^2}\log\frac{\xi_U+\mathrm{i}c}{\xi_L+\mathrm{i}c}+\frac{\xi_U-\xi_L}{(\xi_U-\zeta)(\xi_L-\zeta)}\left(\frac{\zeta^2+1}{\zeta+\mathrm{i}c}-\frac{z_0}{R}\right)\right.$$
$$\left.+\ \frac{\zeta^2+2\mathrm{i}c\zeta-1}{(\zeta+\mathrm{i}c)^2}\log\frac{\xi_U-\zeta}{\xi_L-\zeta}\right]. \quad (19)$$

Therefore, the stress function $\phi(\xi)$ can be obtained immediately by substituting the expressions for $\Im(\zeta)$ and $\Im'(\zeta)$ in Equation (13). On inserting all pertaining functions into Equation (14) and applying the Cauchy formulae again, one can write the other stress function ψ as

$$\psi(\zeta) = -\Im_0(\zeta)+\frac{(\zeta^2+1)(\zeta+\mathrm{i}c)^2\,\Im'(\zeta)}{\zeta(1-\mathrm{i}c\zeta)(\zeta^2+2\mathrm{i}c\zeta-1)}-\frac{(1-c^2)^2\,\Im'(1/\mathrm{i}c)}{\mathrm{i}c(1-\mathrm{i}c\zeta)}$$
$$-\ \frac{(1-c^2)(c^2\,\Im'(1/\mathrm{i}c)-\overline{\Im'(1/\mathrm{i}c)})(\zeta^2+2\mathrm{i}c\zeta+1)}{(1+c^2)\zeta(\zeta^2+2\mathrm{i}c\zeta-1)}, \quad (20)$$

where $\Im_0(\zeta)$, denoting the Cauchy integral of $\overline{f}(\xi^{-1})$, is given by

$$\Im_0(\zeta) = \frac{\overline{F}\,R}{2\pi\mathrm{i}}\left[\frac{c^2-1}{c(c\zeta+\mathrm{i})}\log\frac{1-\mathrm{i}c\,\xi_U}{1-\mathrm{i}c\,\xi_L}-\frac{1}{\zeta}\log\frac{\xi_U}{\xi_L}\right.$$
$$\left.+\ \left(\frac{\zeta^2+1}{\zeta(1-\mathrm{i}c\zeta)}-\frac{\overline{z_0}}{R}\right)\log\frac{\xi_U-\zeta}{\xi_L-\zeta}\right]. \quad (21)$$

It is worth noting that the first and the third term in the bracket on the right-hand side of Equation (21) have together removed the principal part at the point $\zeta = -\mathrm{i}\,c^{-1}$, and thus $\psi(\zeta)$ is holomorphic throughout the whole plane outside the unit circle ρ. As a result, the obtained stress functions along with the expressions for $\Im(\zeta)$, $\Im'(\zeta)$ and $\Im_0(\zeta)$, explicitly constitute the complete solution to the partially loaded CAC problem. To evaluate the corresponding stress, as well as displacement field in the domain, consecutive differentiations using the chain rule to evaluate $\mathrm{d}\phi(\zeta)/\mathrm{dz}$, $\mathrm{d}^2\phi(\zeta)/\mathrm{dz}^2$, and $\mathrm{d}\psi(\zeta)/\mathrm{dz}$, are performed and all expressions are formulated in Appendix 1 for reference. Eventually, from the basic complex-variable method, corresponding stresses an arbitrary point can promptly be determined by

$$\begin{aligned}\sigma_{xx} &= 2\,\mathfrak{Re}\{\mathrm{d}\phi/\mathrm{d}z\}-\mathfrak{Re}\{\overline{z}\mathrm{d}^2\phi/\mathrm{d}z^2+\mathrm{d}\psi/\mathrm{d}z\}\\ \sigma_{yy} &= 2\,\mathfrak{Re}\{\mathrm{d}\phi/\mathrm{d}z\}+\mathfrak{Re}\{\overline{z}\,\mathrm{d}^2\phi/\mathrm{d}z^2+\mathrm{d}\psi/\mathrm{d}z\}\\ \sigma_{xy} &= \mathfrak{Im}\{\overline{z}\,\mathrm{d}^2\phi/\mathrm{d}z^2+\mathrm{d}\psi/\mathrm{d}z\}\end{aligned} \quad (22)$$

where the operators $\Re\{\}$ and $\Im\{\}$ represent the real and imaginary parts of the complex variables in the curly brackets. Also, the displacement components can be obtained directly by

$$u_x+\mathrm{i}u_y = (\kappa\phi - z\overline{\phi}' - \overline{\psi})/2G, \quad (23)$$

in which u_x and u_y are the displacement components in the x- and y-directions, respectively; G is the shear modulus and κ is written in terms of the Poisson ratio ν by $\kappa = (3-\nu)/(1+\nu)$

for plane stress problems. For the case of plane strain, Young's modulus E should be modified by replacing E by $E/(1-\nu^2)$ and ν by $\nu/(1-\nu)$. It is worth mentioning that, if ξ_U is equal to ξ_L (by substituting $\theta_0 = \pi$ in Equation (6)), the solution will be reduced to the one to the special problem when the whole crack surface is fully loaded.

3.2. STRESS-INTENSITY FACTORS

Stress-intensity factors, abbreviated as SIFs, were originally introduced to indicate the strength of singularities at crack tips. It was shown [3] that the complex-variable technique by Muskhelishvili [2] could be conveniently incorporated into computing SIFs for various configurations. In the vicinity of the crack tip z_-, SIF can be evaluated directly through

$$\text{SIF} = K_I - \mathrm{i}\, K_{II} = 2\sqrt{2\pi} \lim_{z\to z_-} \sqrt{z-z_-}\frac{\mathrm{d}\phi(z)}{\mathrm{d}z}, \tag{24}$$

where the subscripts *I* and *II* denote fracture mode 1 and mode 2, respectively. In order to evaluate SIFs by use of Equation (24), the coordinate system has to be rotated in such a way that the crack tip of interest is parallel to the x-axis. For this purpose, an expedient transformation is made such that the crack tip of interest is not only tangent to the x-axis but also located at the origin ($z = 0$). This transformation for the left crack tip takes the following form,

$$z = \frac{\hat{z}}{2c^2 - 1 + 2\mathrm{i}c\sqrt{1-c^2}} - a\left(1 + \frac{\mathrm{i}c}{\sqrt{1-c^2}}\right), \tag{25}$$

where $\hat{z}$ is the transformed coordinate system. Due to the unsymmetrical loading, different coordinate transformations are required to relocate each crack tip of interest. For this purpose, the transformation for the right tip takes the following form,

$$z = \frac{\hat{z}}{1 - 2c^2 + 2\mathrm{i}c\sqrt{1-c^2}} + a\left(1 - \frac{\mathrm{i}c}{\sqrt{1-c^2}}\right). \tag{26}$$

By using the chain rule to carry out the differentiation $\mathrm{d}\hat{\phi}/\mathrm{d}\hat{z}$ and going through a lengthy process of taking limits, one may write the SIFs as

$$\begin{aligned}
\text{SIF} &= K_{I_-} - \mathrm{i}\, K_{II_-} = \sqrt{\pi a}\,\frac{\xi_+}{R}\left[-\Im'(\xi_-) + \frac{c^2\,\Im'(-i\,c^{-1}) - \overline{\Im'}(i\,c^{-1})}{1+c^2}\right],\\
\text{SIF} &= K_{I_+} - \mathrm{i}\, K_{II_+} = \sqrt{\pi a}\frac{-\xi_-}{R}\left[-\Im(\xi_+) + \frac{c^2\,\Im'(-i\,c^{-1}) - \overline{\Im'}(i\,c^{-1})}{1+c^2}\right],
\end{aligned} \tag{27}$$

where ξ_- and ξ_+ are coordinates of the left and right crack tip in the ζ, respectively; the function $\Im'$ is given by Equation (19). As long as the loading condition and the configuration of a CAC are specified (z_0, c and a are determined), the stress intensity factors for both tips can be calculated directly through Equation (27) in a straightforward manner.

3.3. EXAMPLE

As an illustration of this partially loading problem, an example case is studied in which only the right half of the crack surface is subjected to uniform tractions. For this special case ($\theta_0 =$

$\pi/2$), ξ_U and ξ_L are equal to $+\mathrm{i}$ and $-\mathrm{i}$, respectively, and the formulations will be significantly simplified. From Equation (19), the constants $\Im'(\xi_-)$, $\Im'(\xi_+)$, and $\Im'(-\mathrm{i}\,c^{-1})$ are found to be

$$\begin{aligned}
\Im'(\xi_-) &= \frac{FR}{2\pi}\left[\pi - \frac{2}{\sqrt{1-c^2}} - \mathrm{i}\log\frac{1+c}{1-c}\right],\\
\Im'(\xi_+) &= \frac{FR}{2\pi}\left[\pi + \frac{2}{\sqrt{1-c^2}} - \mathrm{i}\log\frac{1+c}{1-c}\right],\\
\Im'(-i\,c^{-1}) &= \frac{FR}{2\pi}\left[\frac{-\pi\,c^2}{1-c^2} + \mathrm{i}\left(\frac{2c}{1-c^2} - \log\frac{1+c}{1-c}\right)\right].
\end{aligned} \tag{28}$$

By substituting the above equations in the stress functions, given by Equations (13) and (20), one may calculate the displacement field using Equation (23), which is plotted in Figure 3 for visualization of the deformed crack surfaces half loaded with tension $P(Q = 0)$. After these constants are substituted back in Equation (27), the final expressions for the corresponding SIFs of this half-loading case are written as

$$\begin{aligned}
K_{I\pm} &= \sqrt{\pi a}P\left(\frac{1-c^2}{2} \pm \frac{1}{\pi(1-c^2)} - \frac{c^2\sqrt{1-c^2}}{2(1+c^2)}\right)\\
&+ \sqrt{\pi a}Q\left(\frac{\pm c}{2(1-c^2)} + \frac{c\sqrt{1-c^2}}{\pi(1+c^2)} - \frac{\sqrt{1-c^2}}{\pi(1+c^2)}\log\frac{1+c}{1-c} + \frac{c}{\pi\sqrt{1-c^2}}\right),\\
K_{II\pm} &= \sqrt{\pi a}P\left(\frac{\mp c}{2(1+c^2)}\right) + \sqrt{\pi a}Q\left(\frac{1}{2\sqrt{1-c^2}} \pm \frac{1}{\pi(1+c^2)} \pm \frac{c}{\pi(1+c^2)}\log\frac{1+c}{1-c}\right).
\end{aligned} \tag{29}$$

As in this special case the CAC is reduced to a straight crack ($c = 0$), the stress-intensity factors are reduced to

$$K_{I\pm} = \sqrt{\pi a}P\left(\frac{1}{2} \pm \frac{1}{\pi}\right) \quad , \quad K_{II\pm} = \sqrt{\pi a}Q\left(\frac{1}{2} \pm \frac{1}{\pi}\right), \tag{30}$$

which are in agreement with the results obtained by Paris and Sih [11]. To investigate the effect of geometric configuration on the corresponding SIFs, it is expedient to introduce some geometric functions for both fracture modes given by

$$GF_I = \frac{K_I}{\sqrt{\pi\,a}\,P} \quad , \quad GF_{II} = \frac{K_{II}}{\sqrt{\pi\,a}\,Q}. \tag{31}$$

For the pure-tension case (Q= 0), the geometric functions are

$$GF_{I\pm}(c) = \frac{1-c^2}{2} \pm \frac{1}{\pi(1-c^2)} - \frac{c^2\sqrt{1-c^2}}{2(1+c^2)}, \quad GF_{II\pm}(c) = \frac{\mp c}{2(1+c^2)}. \tag{32}$$

Because the change rate of SIFs due to crack configuration is of primary interest, absolute values of the geometric functions are plotted in Figure 4. As shown in the figure, the SIF of fracture mode I for the left crack tip reaches its critical zero value when c is about 0·377 ($\eta = 22{\cdot}2°$). At this critical value, the extensional effect of the crack opening traction on the left crack tip is counteracted by a compressive effect caused by the curvature of the crack

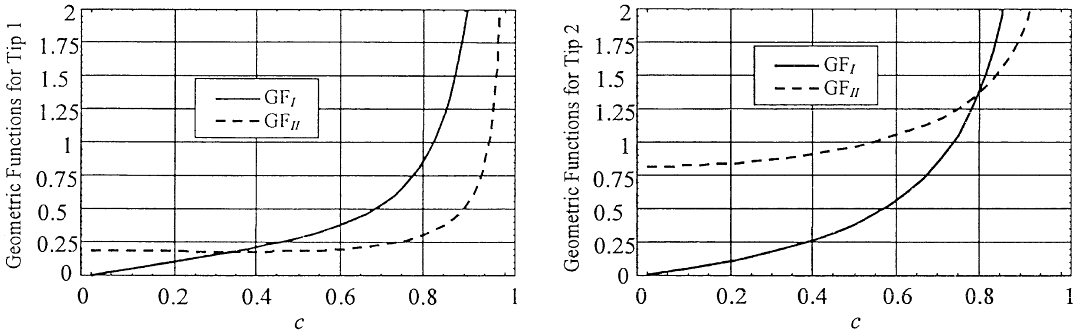

Figure 5 Variation of Geometric Functions with variable c for pure shearing.

surfaces, and, as a result the left crack tip, stays at its neutral position with zero SIF. As c increases beyond this critical value, the corresponding SIF at the left crack tip continues to decline and yields negative values, which imply compression of both crack surfaces to each other near the tip. However, for the right crack tip, the SIF of mode I remains positive and reaches a minimum value of 0·708 when c is about 0·558. In a similar way, the geometric functions for the case of pure shearing ($P = 0$) can also be defined by

$$\begin{aligned} GF_{I\pm} &= \frac{\pm c}{2(1-c^2)} + \frac{c\sqrt{1-c^2}}{\pi(1+c^2)} - \frac{\sqrt{1-c^2}}{\pi(1+c^2)}\log\frac{1+c}{1-c} + \frac{c}{\pi\sqrt{1-c^2}}, \\ GF_{II\pm} &= \frac{1}{2\sqrt{1-c^2}} \pm \frac{1}{\pi(1+c^2)} \pm \frac{c}{\pi(1+c^2)}\log\frac{1+c}{1-c}, \end{aligned} \tag{33}$$

for which absolute values are plotted in Figure 5.

Suppose the same plate is subjected to biaxial tension P at infinity. By the principle of superposition, one may, instead, investigate the equivalent case that an overall uniform traction P is applied along the whole crack surface. For such a case, by letting θ_0 equal π in Equation (7), corresponding SIFs can be readily shown to be

$$K_I = \sqrt{\pi a}\,\frac{\sqrt{1-c^2}}{1+c^2}\,P \quad , \quad K_{II} = \sqrt{\pi a}\,\frac{c}{1+c^2}\,P, \tag{34}$$

which is, in fact, identical with the result given by Sih *et al.* [3]. In the same way, the geometric functions for this biaxial-tension case can be defined by

$$GF_{I\pm} = \frac{\sqrt{1-c^2}}{(1+c^2)} \quad , \quad GF_{II\pm} = \frac{c}{(1+c^2)}, \tag{35}$$

which are plotted in Figure 6. Once the solution to the partial-loading problem is obtained, one may proceed to investigate the special case that concentrated loads are applied on the crack edge. The solution to this special case is considered useful, since it may be taken to formulate the solution to the CAC problem under general loading conditions.

4. Concentrated loads applied on the surface of a CAC

The special object of interest in this paper is to formulate the solution to the problem when two pairs of concentrated loads, consisting of the crack opening forces P' (in the outward normal

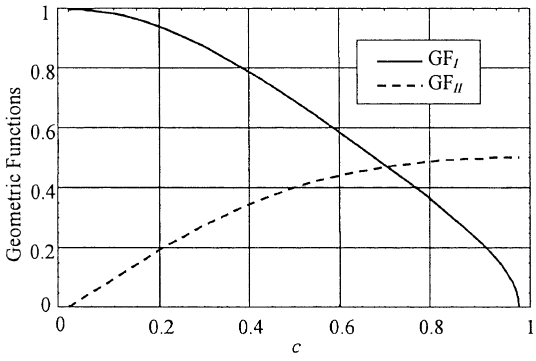

Figure 6. Geometric functions as a function of c for the biaxial tension case.

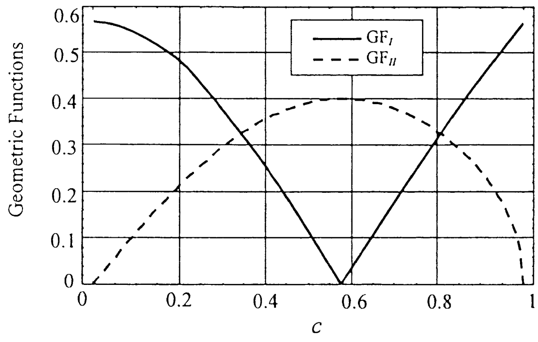

Figure 7. Geometric functions as a function of c for concentrated loads applied in the middle.

direction) and Q' (in the clockwise direction), act on both of the upper and lower surface of a CAC as shown in Figure 1. Suppose an arbitrary point in the ζ-plane is oriented with an angle Θ measured from the reference line. From Equation (6), the coordinate of this point $>_S$ is given by

$$\xi_S = (c \sin\Theta + \sqrt{1 - c^2 \cos^2\Theta})\, \mathrm{e}^{\mathrm{i}\Theta} - \mathrm{i}c. \tag{36}$$

Therefore, by substituting the expression for ξ_S in the mapping function given by Equation (7), the corresponding point of ξ_S in the z-plane, denoted by z_S, is expressed as

$$z_S = 2R\cos\Theta\left(\sqrt{1 - c^2\cos^2\Theta} - \mathrm{i}c\cos\Theta\right). \tag{37}$$

For the present problem, suppose one set of concentrated loads, P' and Q', act upon the upper surface (with a reference angle Θ), while the same amount of concentrated loads, *i.e.*, P' and Q', act in the opposite direction upon the lower surface (with a reference angle $-\Theta$). Before formulating concentrated loads on the crack surface, one has to prescribe uniform tractions over a differential increment of the crack surface. For this purpose, Equation (37) is differentiated with respect to Θ to give

$$\mathrm{d}z_S = \frac{-2R\sin\Theta\left(c\cos\Theta + \mathrm{i}\sqrt{1 - c^2\cos^2\Theta}\right)^2}{\sqrt{1 - c^2\cos^2\Theta}}\mathrm{d}\Theta. \tag{38}$$

Suppose the uniform traction given by Equation (15) acts upon an infinitesimal increment $\mathrm{d}z_s$. Obviously, a concentrated force, Γ, can be formulated by prescribing uniform traction F over $\mathrm{d}z_s$. Thus, the concentrated force can be expressed as

$$\Gamma = -P' + \mathrm{i}Q' = F\mathrm{d}z_S = F\frac{-2R\sin\Theta\left(c\cos\Theta + \mathrm{i}\sqrt{1 - c^2\cos^2\Theta}\right)^2}{\sqrt{1 - c^2\cos^2\Theta}}\mathrm{d}\Theta. \tag{39}$$

Thus, an infinitesimal increment of Θ can be obtained by

$$\mathrm{d}\Theta = \frac{\Gamma}{F}\frac{\sqrt{1 - c^2\cos^2\Theta}}{-2R\sin\Theta\left(c\cos\Theta + \mathrm{i}\sqrt{1 - c^2\cos^2\Theta}\right)^2} \tag{40}$$

Principally, the general complex stress functions of this problem are exactly of the same form as those for the partially loaded CAC, *i.e.*, Equations (13) and (20). However, some modifications for the terms $\Im(\zeta)$ and $\Im_0(\zeta)$ to consider the concentrated loads are necessary. For this purpose, all pertaining functions, $\Im(\zeta)$, $\Im'(\zeta)$, and $\Im_0(\zeta)$, are differentiated with respect to Θ and multiplied by $d\Theta$. Through a lengthy process of differentiations and rearrangements of terms, $\Im(\zeta)$ and $\Im_0(\zeta)$ can be expressed by

$$\begin{aligned} \Im(\zeta) &= \frac{-\Gamma}{2\pi \mathrm{i}\Delta_1}\left\{\frac{c^2-1}{(\zeta+\mathrm{i}c)}\,\Delta_2+\left(\frac{\zeta^2+1}{\zeta+ic}-\frac{z_S}{R}\right)\Delta_3-\log\frac{\xi_U-\zeta}{\xi_L-\zeta}\,\Delta_1\right\}, \\ \Im_0(\zeta) &= \frac{\overline{-\Gamma}}{2\pi\,\mathrm{i}\,\overline{\Delta_1}}\left\{\frac{c^2-1}{c(c\zeta+\mathrm{i})}\,\Delta_4-\frac{\Delta_5}{\zeta}+\left(\frac{\zeta^2+1}{\zeta(1-\mathrm{i}c\zeta)}-\frac{\overline{z_S}}{R}\right)\Delta_3-\log\frac{\xi_U-\zeta}{\xi_L-\zeta}\,\Delta_6\right\}, \end{aligned} \quad (41)$$

where ξ_U and ξ_L are given by Equations (6) with θ_0 replaced by Θ and $\Delta_1 \sim \Delta_6$ are defined by

$$\begin{aligned} \Delta_1 &= \frac{{\xi_U}^2+2\mathrm{i}c\,\xi_U-1}{(\xi_U+\mathrm{i}c)^2}\xi'_U \quad, \ \Delta_2=\left(\frac{\xi'_U}{\xi_U+\mathrm{i}c}-\frac{\xi'_L}{\xi_L+\mathrm{i}c}\right), \\ \Delta_3 &= \left(\frac{\xi'_U}{\xi_U-\zeta}-\frac{\xi'_L}{\xi_L-\zeta}\right) \ , \ \Delta_4=\mathrm{i}c\left(\frac{\xi'_L}{1-\mathrm{i}c\,\xi_L}-\frac{\xi'_U}{1-\mathrm{i}c\,\xi_U}\right), \\ \Delta_5 &= \left(\frac{\xi'_U}{\xi_U}-\frac{\xi'_L}{\xi_L}\right) \qquad , \ \Delta_6=\frac{\xi'_U({\xi_U}^2+2\mathrm{i}c\,\xi_U-1)}{{\xi_U}^2\,(1-\mathrm{i}c\,\xi_U)^2} \end{aligned} \quad (42)$$

and ξ'_U and ξ'_L are

$$\xi'_U = \left(c+\mathrm{i}\frac{\mathrm{e}^{-i\Theta}-c^2\cos\Theta}{\sqrt{1-c^2\cos^2\Theta}}\right)\mathrm{e}^{2\mathrm{i}\Theta}, \qquad \xi'_L=-\left(c+\mathrm{i}\frac{\mathrm{e}^{i\Theta}-c^2\cos\Theta}{\sqrt{1-c^2\cos^2\Theta}}\right)\mathrm{e}^{-2\mathrm{i}\Theta}. \quad (43)$$

Similarly, the functions $\Im'(\zeta)$, $\Im''(\zeta)$ and $\Im'_0(\zeta)$ can also be obtained through direct differentiation and the final simplified expressions for these functions are listed in Appendix 1 for reference. Thus, the expressions for the stress functions accompanied with the functions $\Im'(\zeta)$, $\Im''(\zeta)$, $\Im'''(\zeta)$, $\Im_0(\zeta)$ and $\Im'_0(\zeta)$ completely construct the general solution to the problem for concentrated loading. Calculations for the stress distributions around the crack surface have been carried out, and it was numerically proved that all boundary conditions on the crack surface are satisfied. Equation (27) is still valid for evaluating SIFs for this problem when the function $\Im'(\zeta)$ for concentrated loads is inserted into the equation. When this curved crack is degenerated into a straight slit ($c = 0$), it can be proved that the reduced solution is actually identical with the one obtained by Sih *et al.* [3].

4.1. Example

Final expressions for the complete solution to this problem of concentrated loads have been built into computer codes to compute SIFs, as well as all displacement and stress components. As an example to illustrate the change rate of SIFs on account of crack configurations, a pair of extensional forces $P'(Q' = 0)$ is assumed to act on the middle surface of a CAC ($\Theta = \pi/2$). The associated geometric functions are plotted in Figure 7 for their absolute values. As shown in the figure, the SIF for mode I keeps declining with c until it reaches its critical zero value when c equals $1/\sqrt{3}$ ($\eta = 35{\cdot}2^0$). As c goes beyond this critical point, the SIF values of mode

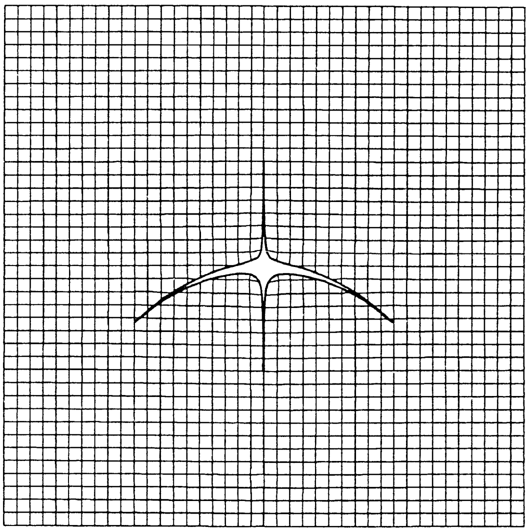

Figure 8. Deformation of a plate containing a CAC subjected to concentrated loads ($c = 0{\cdot}3$).

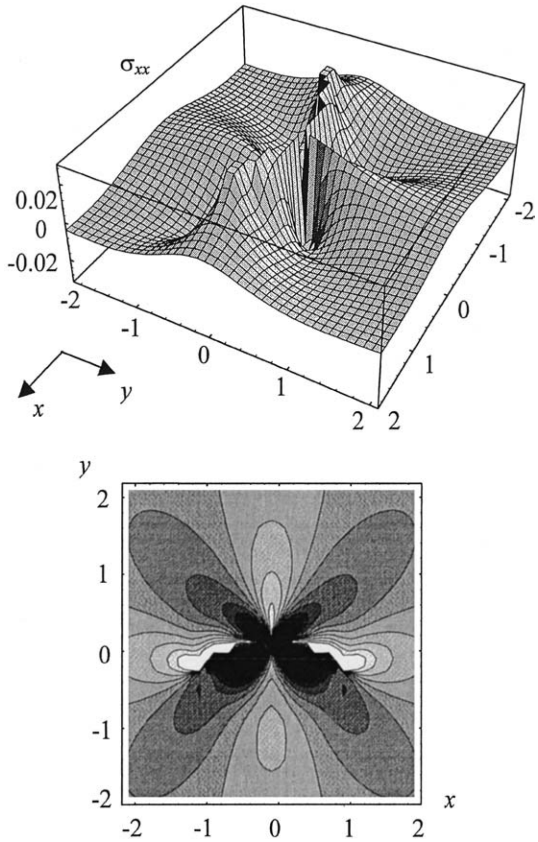

Figure 9. Distribution of σ_{xx} in the vicinity of a CAC subjected to concentrated loads ($c = 0{\cdot}3$).

I will become negative due to the compression caused by the concentrated load applied on the lower crack surface. In contrast, the SIF for mode II starts climbing up from $c = 0$ (for a straight cut), until it reaches a maximum value of 0·3989 when c hits the critical value of $1/\sqrt{3}$.

As an illustration, a sample thin plate with $c = 0{\cdot}3$, $a = 1$ (unit) and the mechanical properties $\nu = 0{\cdot}33$, $E = 10$ (units) are assumed for computing all displacements and stress components in the vicinity of the CAC. To visualize the obtained solution, all displacement components are numerically computed by use of Equation (23) and the displaced mesh grids are plotted in Figure 8. In this figure, two spikes present on the mid-surfaces of CAC are due to the singularities at the points where concentrated loads are applied. Also, all stress components σ_{xx}, σ_{xx}, and $\sigma_{<ityy}$ corresponding to the obtained complex stress functions are calculated in the vicinity of CAC and plotted in Figure 9, Figure 10, and Figure 11, respectively.

4.2. Arbitrary loading

The purpose of investigating the special case when the concentrated loads act on the upper and lower surface of an arbitrary point at a CAC is to formulate the solution to the general loading problem. From the principle of superposition, one may construct the solution to the arbitrary

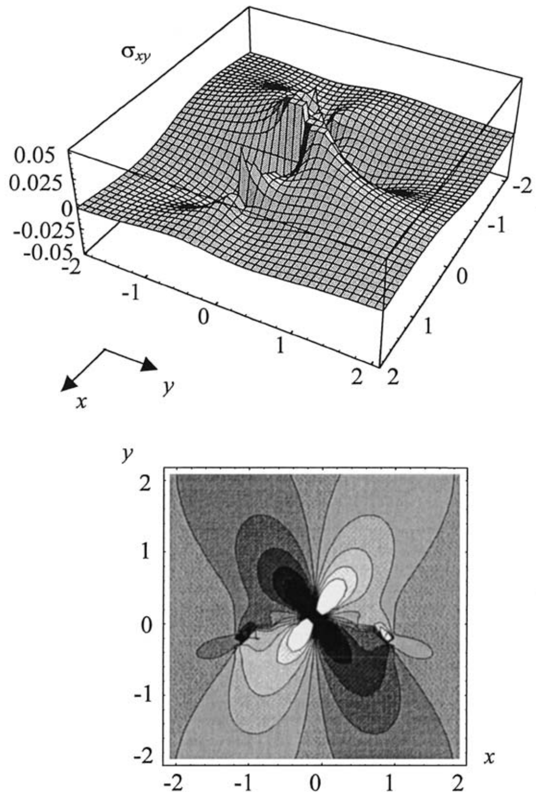

Figure 10. Distribution of σ_{xy} in the vicinity of a CAC subjected to concentrated loads ($c = 0{\cdot}3$).

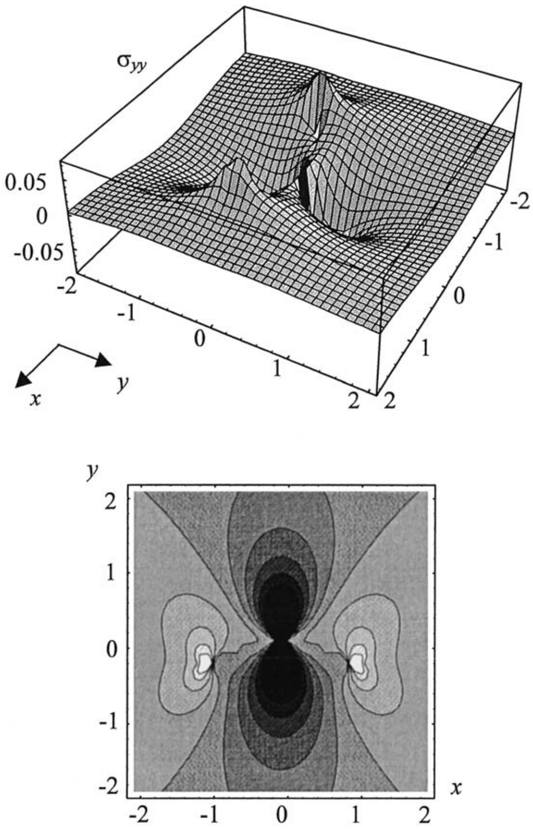

Figure 11. Distribution of σ_{yy} in the vicinity of a CAC subjected to concentrated loads ($c = 0{\cdot}3$).

loading problem by replacing Γ with $\Lambda(\Theta) \cong \mathrm{d}z_S$, where $\Lambda(\Theta)$ is the crack opening traction around the circumference of the crack to yield a traction-free condition along the crack edge. Therefore, the Cauchy integral of the boundary stress integration functions can be written as

$$\begin{aligned} \Im(\zeta) &= \frac{R}{2\pi \mathrm{i}} \int_0^{\pi} \Lambda(\Theta) \left\{ \frac{c^2 - 1}{(\zeta + \mathrm{i}c)} \Delta_2 + \left(\frac{\zeta^2 + 1}{\zeta + \mathrm{i}c} - \frac{z_S}{R} \right) \Delta_3 - \log \frac{\xi_U - \zeta}{\xi_L - \zeta} \Delta_1 \right\} \mathrm{d}\Theta, \\ \Im_0(\zeta) &= \frac{R}{2\pi \mathrm{i}} \int_0^{\pi} \overline{\Lambda(\Theta)} \left\{ \frac{c^2 - 1}{c(c\zeta + \mathrm{i})} \Delta_4 - \frac{\Delta_5}{\zeta} + \left(\frac{\zeta^2 + 1}{\zeta(1 - \mathrm{i}c\zeta)} - \frac{\overline{z_S}}{R} \right) \Delta_3 - \log \frac{\xi_U - \zeta}{\xi_L - \zeta} \Delta_6 \right\} \mathrm{d}\Theta. \end{aligned} \tag{44}$$

In the polar coordinate system (r,θ), with the origin sitting at the arc center, the crack opening stress $\Lambda(\Theta)$, varying along the circumference of the crack, can be further expressed by $\Lambda(\Theta) = -\sigma_{rr}(\Theta) + \mathrm{i}\sigma_{r\theta}\ (\Theta)$. As long as corresponding radial and tangential stresses are determined for a non-cracked plate subjected to arbitrary loads, Equations (44) can be evaluated directly without difficulty using any standard numerical integration scheme. Thereafter, by inserting the obtained numerical values back into the formulations for the associated stress functions, one may calculate all displacements and stresses at an arbitrary point in the domain. Thus, Equations (13), (20) and (44) together constitute the general solution to the CAC problem for arbitrary loading.

5. Conclusions

By the complex-variable technique, the present work provides an alternative crack model to deal with a circular-arc crack subjected to arbitrary loading. In this paper, a conformal-mapping function is applied to transform the contour surface of a circular-arc crack to a unit circle in the auxiliary plane. Through this mapping function, the solution to the problem of a partially loaded CAC is formulated. The solution is further modified to consider the special case where two pairs of crack-opening concentrated loads act upon the crack surface. By the principle of superposition, the modified solution to take account of concentrated loads is used to formulate the general solution to the CAC problem under arbitrary loading conditions. As a matter of fact, the approach involving this mapping has actually integrated the solutions to both the CAC and the straight cut problem into one unique form. In this paper, all reduced solutions when CAC is degenerated into a straight slit are proved to be in agreement with those reported in the literature. The approach adopted in this paper provides an expedient tool for investigating the CAC problem.

Appendix, Functions needed for calculations and displacements

For convenience of engineers' use, the following formulations are supplied for all pertaining functions needed for the calculation of displacements and stress components for the CAC problems described in previous literature:

$$\phi(\zeta) = -\Im(\zeta) - \frac{1-c^2}{1+c^2}\frac{c^2\,\Im'(-\mathrm{i}\,c^{-1}) - \overline{\Im}'(\mathrm{i}\,c^{-1})}{\zeta + \mathrm{i}c}, \tag{A1}$$

$$\frac{\mathrm{d}\phi(\zeta)}{\mathrm{d}z} = \frac{-\Im'(\zeta)\,(\zeta+\mathrm{i}c)^2}{R(\zeta^2+2\mathrm{i}c\zeta-1)} + \frac{1-c^2}{1+c^2}\frac{c^2\,\Im'(-\mathrm{i}\,c^{-1}) - \overline{\Im}'(\mathrm{i}\,c^{-1})}{R(\zeta^2+2\mathrm{i}c\zeta-1)}, \tag{A2}$$

$$\begin{aligned}\frac{\mathrm{d}^2\phi(\zeta)}{\mathrm{d}z^2} &= \frac{(\zeta+\mathrm{i}c)^3}{R^2\,(\zeta^2+2\mathrm{i}c\zeta-1)^3}\Big\{2(1-c^2)\Im'(\zeta) - (\zeta+\mathrm{i}c)(\zeta^2+2\mathrm{i}c\zeta-1)\Im''(\zeta)\\ &\quad - \frac{2(1-c^2)[c^2\,\Im'(-\mathrm{i}\,c^{-1}) - \overline{\Im}(\mathrm{i}\,c^{-1})]}{(1+c^2)}\Big\},\end{aligned} \tag{A3}$$

$$\begin{aligned}\psi(\zeta) &= -\Im_0(\zeta) + \frac{(\zeta^2+1)\,(\zeta+\mathrm{i}c)^2\,\Im'(\zeta)}{\zeta(1-\mathrm{i}c\zeta)(\zeta^2+2\mathrm{i}c\zeta-1)} - \frac{(1-c^2)^2\,\Im'(1/\mathrm{i}c)}{\mathrm{i}c(1-\mathrm{i}c\zeta)}\\ &\quad - \frac{(1-c^2)(c^2\,\Im'(1/\mathrm{i}c) - \overline{\Im'(1/\mathrm{i}c)})(\zeta^2+2\mathrm{i}c\zeta+1)}{(1+c^2)\zeta(\zeta^2+2\mathrm{i}c\zeta-1)},\end{aligned} \tag{A4}$$

$$\frac{\mathrm{d}\psi(\zeta)}{\mathrm{d}z} = \frac{(\zeta+\mathrm{i}c)^2\,\psi'(\zeta)}{R(\zeta^2+2\mathrm{i}c\zeta-1)}, \tag{A5}$$

$$\begin{aligned}\psi'(\zeta) \;=\; & -\Im_0'(\zeta) - \frac{(1-c^2)^2\,\Im(-i\,c^{-1})}{(1-\mathrm{i}c\zeta)^2} + [\zeta^4 + 4\mathrm{i}c\,\zeta^3 + (4-4\,c^2)\,\zeta^2 + 4\mathrm{i}c\zeta - 1] \\ & \frac{(1-c^2)[c^2\,\Im'(-i\,c^{-1}) - \overline{\Im}'(i\,c^{-1})]}{\zeta^2\,(\zeta^2+2\mathrm{i}c\zeta-1)^2(1+c^2)} + \zeta^{-2}\,(1-\mathrm{i}c\zeta)^{-2}\,(\zeta^2+2\mathrm{i}c\zeta-1)^{-2} \\ \times\; & [\zeta^6 - (2\,\mathrm{i}c^3 - 8\mathrm{i}c)\,\zeta^5 + (2\,c^4 - 13\,c^2 - 4)\,\zeta^4 - (12\,\mathrm{i}c^3 + 8\mathrm{i}c)\,\zeta^3 \\ & + (6\,c^4 + 10\,c^2 - 1)\,\zeta^2 + 6\,\mathrm{i}c^3\,\zeta - c^2]\Im'(\zeta) + \zeta^{-1}\,(1-\mathrm{i}c\zeta)^{-2}\,(\zeta^2+2\mathrm{i}c\zeta-1)^{-2} \\ \times\; & [-\mathrm{i}c\,\zeta^7 + (1+4\,c^2)\,\zeta^6 + (5\,\mathrm{i}c^3 + 4\mathrm{i}c)\,\zeta^5 - (3\,c^2 + 2\,c^4)\,\zeta^4 + (3\mathrm{i}c + 2\,\mathrm{i}c^3)\,\zeta^3 \\ & - (1+6\,c^2+2\,c^4)\,\zeta^2 - (2\mathrm{i}c + 3\,\mathrm{i}c^3)\zeta + c^2]\Im''(\zeta). \end{aligned} \tag{A6}$$

(1) For partial loading:

$$\Im(\zeta) = \frac{FR}{2\pi\mathrm{i}}\left[\frac{c^2-1}{\zeta+\mathrm{i}c}\cdot\log\frac{\xi_U+\mathrm{i}c}{\xi_L+\mathrm{i}c} + \left(\frac{\zeta^2+1}{\zeta+\mathrm{i}c} - \frac{z_0}{R}\right)\log\frac{\xi_U-\zeta}{\xi_L-\zeta}\right], \tag{A7}$$

$$\begin{aligned}\Im'(\zeta) \;=\; & \frac{FR}{2\pi\mathrm{i}}\left[\frac{1-c^2}{(\zeta+\mathrm{i}c)^2}\log\frac{\xi_U+\mathrm{i}c}{\xi_L+\mathrm{i}c} + \frac{\xi_U-\xi_L}{(\xi_U-\zeta)(\xi_L-\zeta)}\left(\frac{\zeta^2+1}{\zeta+\mathrm{i}c} - \frac{z_0}{R}\right)\right. \\ + & \left.\frac{\zeta^2+2\mathrm{i}c\zeta-1}{(\zeta+\mathrm{i}c)^2}\log\frac{\xi_U-\zeta}{\xi_L-\zeta}\right], \end{aligned} \tag{A8}$$

$$\begin{aligned}\Im''(\zeta) \;=\; & \frac{FR}{2\pi\mathrm{i}}\left\{\frac{2(c^2-1)}{(\zeta+\mathrm{i}c)^3}\log\frac{(\xi_U+\mathrm{i}c)(\xi_L-\zeta)}{(\xi_L+\mathrm{i}c)(\xi_U-\zeta)} + \frac{(\xi_U-\xi_L)(\xi_U+\xi_L-2\zeta)}{(\xi_U-\zeta)^2\,(\xi_L-\zeta)^2}\left(\frac{\zeta^2+1}{\zeta+\mathrm{i}c} - \frac{z_0}{R}\right)\right. \\ + & \left.\frac{2(\xi_U-\xi_L)(\zeta^2+2\mathrm{i}c\zeta-1)}{(\xi_U-\zeta)(\xi_L-\zeta)((\zeta+\mathrm{i}c)^2)}\right\}, \end{aligned} \tag{A9}$$

$$\Im_0(\zeta) \;=\; \frac{\overline{F}\,R}{2\pi\mathrm{i}}\left\{\frac{c^2-1}{c(c\zeta+\mathrm{i})}\log\frac{1-\mathrm{i}c\,\xi_U}{1-\mathrm{i}c\,\xi_L} - \frac{1}{\zeta}\log\frac{\xi_U}{\xi_L} + \left(\frac{\zeta^2+1}{\zeta(1-\mathrm{i}c\zeta)} - \frac{\overline{z_0}}{R}\right)\log\frac{\xi_U-\zeta}{\xi_L-\zeta}\right\}, \tag{A10}$$

$$\begin{aligned}\Im_0'(\zeta) \;=\; & \frac{\overline{F}\,R}{2\pi\,\mathrm{i}}\left\{\frac{1-c^2}{(c\zeta+\mathrm{i})^2}\log\frac{1-\mathrm{i}c\,\xi_U}{1-\mathrm{i}c\,\xi_L} + \frac{1}{\zeta^2}\log\frac{\xi_U}{\xi_L} + \frac{\zeta^2+2\mathrm{i}c\zeta-1}{\zeta^2\,(1-\mathrm{i}c\zeta)^2}\log\frac{\xi_U-\zeta}{\xi_L-\zeta}\right. \\ + & \left.\frac{\xi_U-\xi_L}{(\xi_U-\zeta)(\xi_L-\zeta)}\left(\frac{\zeta^2+1}{\zeta(1-\mathrm{i}c\zeta)} - \frac{\overline{z_0}}{R}\right)\right\} \end{aligned} \tag{A11}$$

(2) For concentrated loads:

$$\Im(\zeta) \;=\; \frac{-\Gamma}{2\pi\mathrm{i}\,\Delta_1}\left\{\frac{c^2-1}{(\zeta+\mathrm{i}c)}\,\Delta_2 + \left(\frac{\zeta^2+1}{\zeta+\mathrm{i}c} - \frac{z_S}{R}\right)\Delta_3 - \log\frac{\xi_U-\zeta}{\xi_L-\zeta}\,\Delta_1\right\}, \tag{A12}$$

$$\Im'(\zeta) \;=\; \frac{-\Gamma}{2\pi\,\mathrm{i}\,\Delta_1}\left\{\frac{1-c^2}{(\zeta+\mathrm{i}c)^2}\,\Delta_2 + \left(\frac{\zeta^2+1}{\zeta+\mathrm{i}c} - \frac{z_S}{R}\right)\Delta_7 - \frac{\xi_U-\xi_L}{(\xi_U-\zeta)}\,\Delta_1 + \frac{\zeta^2+2\,i\,c\zeta-1}{(\zeta+i\,c)^2}\,\Delta_3\right\}, \tag{A13}$$

$$\Im''(\zeta) = \frac{-\Gamma}{2\pi \mathrm{i}\Delta_1}\left\{\frac{2(c^2-1)}{(\zeta+\mathrm{i}c)^3}\Delta_2+\left(\frac{\zeta^2+1}{\zeta+\mathrm{i}c}-\frac{z_S}{R}\right)\Delta_8-\frac{(\xi_U-\xi_L)(\xi_U+\xi_L-2\zeta)}{(\xi_U-\zeta)^2(\xi_L-\zeta)^2}\Delta_1\right.$$
$$\left.+\frac{2(\zeta^2+2\mathrm{i}c\zeta-1)}{(\zeta+\mathrm{i}c)^2}\Delta_7+\frac{2(1-c^2)}{(\zeta+\mathrm{i}c)^3}\right\}, \quad \text{(A14)}$$

$$\Im_0(\zeta) = \frac{\overline{-\Gamma}}{2\pi \mathrm{i}\,\Delta_1}\left\{\frac{c^2-1}{c(c\zeta+\mathrm{i})}\Delta_4-\frac{\Delta_5}{\zeta}+\left(\frac{\zeta^2+1}{\zeta(1-\mathrm{i}c\zeta)}-\frac{\overline{z_S}}{R}\right)\Delta_3-\log\frac{\xi_U-\zeta}{\xi_L-\zeta}\Delta_6\right\}, \quad \text{(A15)}$$

$$\Im_0'(\zeta) = \frac{\overline{-\Gamma}}{2\pi \mathrm{i}\,\Delta_1}\left\{\frac{1-c^2}{(c\zeta+i)^2}\Delta_4+\frac{\Delta_5}{\zeta^2}+\left(\frac{\zeta^2+1}{\zeta(1-\mathrm{i}c\zeta)}-\frac{\overline{z_S}}{R}\right)\Delta_7+\frac{\zeta^2+2\mathrm{i}c\zeta-1}{\zeta^2(1-\mathrm{i}c\zeta)^2}\Delta_3\right.$$
$$\left.-\frac{\xi_U-\xi_L}{(\xi_U-\zeta)(\xi_L-\zeta)}\Delta_6\right\}, \quad \text{(A16)}$$

where $\Delta_1 \sim \Delta_8$ are given by

$$\Delta_1 = \frac{\xi_U({\xi_U}^2+2\mathrm{i}c\,\xi_U-1)}{(\xi_U+\mathrm{i}c)^2}, \quad , \quad \Delta_2 = \frac{\xi_U}{\xi_U+\mathrm{i}c}-\frac{\xi_L}{\xi_L+\mathrm{i}c},$$
$$\Delta_3 = \frac{\xi_U}{\xi_U-\zeta}-\frac{\xi_L}{\xi_L-\zeta}, \quad , \quad \Delta_4 = \mathrm{i}c\frac{\xi_L'}{1-\mathrm{i}c\,\xi_L}-\frac{\xi_U'}{1-\mathrm{i}c\,\xi_U},$$
$$\Delta_5 = \frac{\xi_U'}{\xi_U'}-\frac{\xi_L}{\xi_L}, \quad , \quad \Delta_6 = \frac{\xi_U'({\xi_U}^2+2\mathrm{i}c\,\xi_U-1)}{{\xi_U}^2(1-\mathrm{i}c\,\xi_U)^2},$$
$$\Delta_7 = \frac{\xi_U'-\xi_L'}{(\xi_U-\zeta)(\xi_L-\zeta)}-\frac{\xi_U'(\xi_U-\xi_L)}{(\xi_U-\zeta)^2(\xi_L-\zeta)}-\frac{\xi_L'(\xi_U-\xi_L)}{(\xi_U-\zeta)(\xi_L-\zeta)^2},$$
$$\Delta_8 = \frac{2\,\xi_U'(\xi_L-\zeta)-2\,\xi_L'(\xi_U-\zeta)}{(\xi_U-\zeta)^2(\xi_L-\zeta)^2}-2(\xi_U-\xi_L)\frac{\xi_L'(\xi_U-\zeta)^2+\xi_U'(\xi_L-\zeta)^2}{(\xi_U-\zeta)^3(\xi_L-\zeta)^3} \quad \text{(A17)}$$

and ξ_U', ξ_L' are given by

$$\xi_U' = \left(c+\mathrm{i}\frac{\mathrm{e}^{-\mathrm{i}\Theta}-c^2\cos\Theta}{\sqrt{1-c^2\cos^2\Theta}}\right)\mathrm{e}^{2\mathrm{i}\Theta},$$
$$\xi_L' = -\left(c+\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\Theta}-c^2\cos\Theta}{\sqrt{1-c^2\cos^2\Theta}}\right)\mathrm{e}^{-2\mathrm{i}\Theta}. \quad \text{(A18)}$$

Acknowledgements

The authors gratefully acknowledge financial support by the National Science Council of Taiwan, Republic of China (Grant Number: NSC-91-2212-E-035-007).

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